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Nearly realcompact spaces

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Abstract

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A Tychonoff space X is nearly realcompact if $\beta X - \upsilon X$ is dense in $\beta X - X$. Several characterizations of nearly realcompact spaces are given. Also given are sufficient conditions for spaces to be nearly realcompact as well as sum, product and mapping theorems.

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Definitions and preliminaries

We assume that all spaces are Tychonoff. For a function $f: X \to Y$ with $A \subset X$ and $B \subset Y$, we follows the second author's lead in setting $f^{\to}(A) = \{f(x): x \in A\}$ and $f^{\leftarrow}(B) = \{x \in X: f(x) \in B\}$. We denote the real numbers by $\mathbb{R}$, the rational numbers by $\mathbb{Q}$ and the natural numbers by $\mathbb{N}$. The first countably infinite ordinal is denoted by ω and the first uncountable one by ω_1. As usual, we set $C(X) = \{f: X \to \mathbb{R}: f$ is continuous$\}$ and $C^*(X) = \{f \in C(X): f$ is bounded$\}$.

A *zero-set* of X is a set of the form $Z(f) = f^{\leftarrow}\{0\}$ for $f \in C(X)$ and a *cozero-set* is the complement of a zero-set. We write $\mathrm{coz}(f) = X - Z(f)$. The reader can see that if $f: X \to Y$ is continuous and A is a zero-set (respectively cozero-set) of Y, then $f^{\leftarrow}(A)$ is a zero-set (respectively cozero-set) of X. A *z-ultrafilter* on X is a maximal filter in the collection of all zero-sets of X.

A subset A of X is *z-embedded* in X if each zero-set of A is of the form $A \cap Z$ where Z is a zero-set of X and A is *C-embedded* (respectively *C^*-embedded*) in X if for every $f \in C(A)$ (respectively $C^*(A)$) there is $g \in C(X)$ such that $g \restriction A = f$. It is well known [3, 1.1] that cozero-sets of X are z-embedded in X.

Terms which are undefined in this paper can be assumed to have the meanings assigned to them in [9]. We assume that the reader is familiar with the theory of z-ultrafilters as well as with the construction and better-known properties of both the Stone-Čech compactification βX and the Hewitt realcompactification vX of a Tychonoff space X as described in [10].

In [12], Henriksen and Rayburn defined a space X to be *nearly pseudocompact* if $vX - X$ is dense in $\beta X - X$; that is, vX is nearly βX. This gave the first author the idea of defining a space X to be *nearly realcompact* if $\beta X - vX$ is dense in $\beta X - X$; that is, X is nearly vX. Obviously every realcompact space is nearly realcompact. The converse is false. Less obviously, every topologically complete space is nearly realcompact (see Corollary 3.2). Nearly realcompactness is hereditary with respect to cozero-sets but not with respect to closed subsets (even C-embedded zero-sets (see Remark 1.17)). No question of the measurability of cardinals arises in the theory of nearly realcompact spaces. In particular, every discrete space is nearly realcompact (see Corollary 2.1).

Nearly realcompact spaces are much more numerous than realcompact spaces; in fact, the product of $\mathbb{Q}$ with any space is nearly realcompact (see Theorem 1.11). Nonetheless, the hypothesis "nearly realcompact" can substitute for that of "realcompact" in many theorems. For example, the second author notes that this is true of all the theorems in [6] (see [6, 2.7]). For another example, the reader can see that a nearly realcompact pseudocompact space is compact.

1. Conditions equivalent to nearly realcompactness

We give several conditions equivalent to nearly realcompactness, both in the general case and in the case where the space is nowhere locally compact.

We will frequently make use of the following results from elsewhere.

Proposition 1.0 [1, 5.1]. *If P is a cozero-set in X, then $vP = vX - \mathrm{cl}_{vX}(X - P)$.*

Hence $P \cap vX = v(P \cap X)$ for all cozero-sets P of βX. Clearly then every cozero-set in a realcompact space is realcompact.

Proposition 1.1 [2, 3.9]. *If X is a countable union of z-embedded realcompact subspaces, then X is realcompact.*

The following characterization of nearly realcompactness is used frequently in the sequel.

Theorem 1.2. *The following conditions on a space X are equivalent*:

(1) *X is nearly realcompact.*

(2) *If P is a nonrealcompact cozero-set in X, then there is a decreasing sequence $\langle Z_n \colon n \in \omega \rangle$ of nonempty zero-sets of X with $Z_n \subset P$ for all $n \in \omega$ and $\bigcap_{n \in \omega} Z_n = \emptyset$.*

Proof. (1)⇒(2) Let P be a nonrealcompact cozero-set in X and let $G = \beta X - \mathrm{cl}_{\beta X}(X-P)$. Let $p \in vP - P$. By Proposition 1.0, $p \notin \mathrm{cl}_{vX}(X-P)$ and so $p \in G \cap (\beta X - X)$. By (1) there is $q \in G \cap (\beta X - vX)$ and so there is $f \in C(\beta X)$ with $f(q)=0$, $Z(f \restriction X) = \emptyset$ and $f \geq 0$ [9, 3.11.10]. Also there is $g \in C(\beta X)$ with $g(q)=0$, $g=1$ on $\beta X - G$ and $g \geq 0$. For each $n \in \omega$, define

$$Z'_n = \left\{ y \in \beta X : (f+g)(y) \leq \frac{1}{n+2} \right\}$$

and set $Z_n = Z'_n \cap X$. Then $\langle Z_n : n \in \omega \rangle$ is a decreasing sequence of nonempty zero-sets of X with $Z_n \subset P$ and $\bigcap_{n\in\omega} Z_n = \emptyset$.

(2)⇒(1) Let G be open in βX with $G \cap (\beta X - X) \neq \emptyset$ and suppose that $G \cap (\beta X - vX) = \emptyset$. There exists $p \in G \cap (vX - X)$ and there exists a cozero-set P, with $p \in P \subset \mathrm{cl}_{\beta X} P \subset G \subset vX$. Then $p \in P \cap vX = v(P \cap X)$, but $p \notin P \cap X$ and so $P \cap X$ is not realcompact. By (2) there is a decreasing sequence $\langle Z_n : n \in \omega \rangle$ of nonempty zero-sets of X with $Z_n \subset P \cap X$ and $\bigcap_{n\in\omega} Z_n = \emptyset$. Now $\{Z_n : n \in \omega\}$ is a zero-set filter base on X and so is contained in a z-ultrafilter $\mathcal{U}$ and there is $q \in \beta X$ with $\mathcal{U} \to q$. But $q \in \mathrm{cl}_{\beta X} P \subset G \subset vX$, and so $\bigcap_{n\in\omega} Z_n \neq \emptyset$, a contradiction. □

A subset A of X is *relatively pseudocompact* in X if for all $f \in C(X)$, $f \restriction A \in C^*(A)$; that is, every continuous function on X is bounded on A. Clearly if A is relatively pseudocompact in X, then so is every subset of A. It is also easy to see that if A is relatively pseudocompact in X and if $f: X \to Y$ is continuous, then $f^{\to}(A)$ is relatively pseudocompact in Y.

We will need the following characterizations of relatively pseudocompact subsets.

Proposition 1.3 [5, 2.6]. *If $A \subset X$, then the following are equivalent:*

(1) *A is relatively pseudocompact in X.*

(2) *For every discrete sequence $\langle U_n : n \in \omega \rangle$ of open subsets of X, $U_n \cap A = \emptyset$ for some $n \in \omega$.*

(3) *$\mathrm{cl}_{vX} A$ is compact.*

A subset $A \subset X$ is *regular closed* in X if $A = \mathrm{cl}_X \, \mathrm{int}_X A$.

Proposition 1.4. *If G is open in X, then the following are equivalent:*

(1) *G is relatively pseudocompact in X.*

(2) *If $\langle Z_n : n \in \omega \rangle$ is a decreasing sequence of nonempty zero-sets in X with $Z_n \subset G$ for all $n \in \omega$, then $\bigcap_{n\in\omega} Z_n \neq \emptyset$.*

(3) *If $f \in C(X)$ with $X - G \subset Z(f)$, then $f \restriction G$ is bounded.*

(4) *If $\langle F_n : n \in \omega \rangle$ is a decreasing sequence of regular closed subsets of X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$, then $\bigcap_{n\in\omega} F_n \neq \emptyset$.*

Proof. (1)$\Rightarrow$(2) Suppose there is a decreasing sequence $\langle Z_n: n \in \omega\rangle$ of nonempty zero-sets in X with $Z_n \subset G$ for all $n \in \omega$ and $\bigcap_{n\in\omega} Z_n = \emptyset$. For each n, choose $f_n \in C(X)$ with $0 \leqslant f_n \leqslant 1$ with $Z_n = Z(f_n)$, and let $f = \sum_{n\in\omega} 2^{-n} f_n$. Then $Z(f) = \bigcap_{n\in\omega} Z_n = \emptyset$, and so $g = 1/f \in C(X)$. Now for all $m \in \omega$ pick $x_m \in Z_m$ and note that $x_m \in Z_k$ for $k \leqslant m$ and so $f(x_m) \leqslant \sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m}$. Then $g(x_m) \geqslant 2^m$. Since for all $m \in \omega$, $x_m \in G$, g is unbounded on G.

(2)$\Rightarrow$(3) Suppose there is $f \in C(X)$ with $X - G \subset Z(f)$ and $f \restriction G$ unbounded. For each $n \in \omega$ let $Z_n = \{x \in X: |f(x)| \geqslant n+1\}$. Then $Z_n \subset G$ and $\langle Z_n: n \in \omega\rangle$ is a decreasing sequence of nonempty zero-sets in X with $\bigcap_{n\in\omega} Z_n = \emptyset$.

(3)$\Rightarrow$(4) Suppose (4) is false. Let $\langle F_n: n \in \omega\rangle$ be a decreasing sequence of regular closed subsets of X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$ and $\bigcap_{n\in\omega} F_n = \emptyset$. Let $V_n = (\operatorname{int}_X F_n) - F_{n+1}$. Let $J = \{n \in \omega: V_n \cap G \neq \emptyset\}$. We claim that $|J| = \omega$. If not, there is $m \in \omega$ such that for all $n \geqslant m$, $\emptyset = V_n \cap G = (\operatorname{int}_X F_n - F_{n+1}) \cap G = (G \cap \operatorname{int}_X F_n) - (G \cap F_{n+1})$ and so $G \cap F_n \subset G \cap F_{n+1}$. Then since the F_n are decreasing, $G \cap F_n = G \cap F_{n+1}$ for all $n \geqslant m$. Then $\emptyset \neq G \cap F_m \subset \bigcap_{n\in\omega} F_n = \emptyset$, a contradiction.

For all $n \in J$, pick $x_n \in V_n \cap G$ and choose $f_n \in C(X)$ with $f_n(x_n) = n$ and $f_n = 0$ on $X - (V_n \cap G)$. Now $\langle V_n: n \in \omega\rangle$ is locally finite in X since $\bigcap_{n\in\omega} F_n = \emptyset$ and so $f = \sum_{n\in\omega} f_n \in C(X)$. Also $X - G \subset Z(f)$, but $f \restriction G$ is unbounded, and so (3) fails.

(4)$\Rightarrow$(1) Suppose (1) fails. Let $f \in C(X)$ with $f \geqslant 0$ and f unbounded on G. For each $n \in \omega$ let $F_n = \operatorname{cl}_X\{x \in X: f(x) > n\}$. Then $\langle F_n: n \in \omega\rangle$ is a decreasing sequence of regular closed sets in X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$ and $\bigcap_{n\in\omega} F_n = \emptyset$. $\square$

Proposition 1.5. *If P is a relatively pseudocompact cozero-set of X, then P is realcompact if and only if P is nearly realcompact.*

Proof. Assume that P is a cozero-set of X that is nearly realcompact but not realcompact. Let $P = \operatorname{coz}(f)$ with $f \geqslant 0$ and let $P_n = f^{\leftarrow}(2^{-n}, \infty)$. Since $P = \bigcup_{n\in\omega} P_n$ and each P_n is a cozero-set, there is, by Proposition 1.1, $m \in \omega$ with P_m not realcompact. By Theorem 1.2, there is a decreasing sequence $\langle Z_n: n \in \omega\rangle$ of zero-sets of P with $Z_n \subset P_m$ for all $n \in \omega$ and $\bigcap_{n\in\omega} Z_n = \emptyset$. Now P is z-embedded in X and so each $Z_n = Y_n \cap P$ where Y_n is a zero-set of X. For all $n \in \omega$ let $Z'_n = Y_n \cap f^{\leftarrow}[2^{-m}, \infty)$. Then $\langle Z'_n: n \in \omega\rangle$ is a decreasing sequence of nonempty zero-sets of X with $Z'_n \subset P$ and $\bigcap_{n\in\omega} Z'_n = \emptyset$, and P is therefore not relatively pseudocompact by Proposition 1.4. $\square$

Corollary 1.6. *If P is a relatively pseudocompact nearly realcompact cozero-set of X, then P is a cozero-set of βX.*

Proof. We may assume that $P = \operatorname{coz}(f) \cap X$ where $f \in C^*(\beta X)$. We show that $P = \operatorname{coz}(f)$. Let $f(p) = \varepsilon > 0$. Then $p \notin \operatorname{cl}_{\upsilon X}(X - P)$ and so, by Proposition 1.0, $p \in \upsilon P = P$ by Proposition 1.5. $\square$

We denote the locally compact part of a space X by X_{lc}; that is, $X_{lc}=\{x\in X: x$ has a compact neighborhood$\}$. If $X_{lc}=\emptyset$, we say that X is *nowhere locally compact.* The reader can see that $X_{lc}=\mathrm{int}_{\beta X}X$. We will see that local compactness (as well as relative pseudocompactness) plays a major role in the theory of nearly realcompact spaces.

Theorem 1.7. *The following are equivalent for any space X:*

(1) *X is nearly realcompact.*

(2) *If Z_0 and Z_1 are disjoint zero-sets of X with Z_1 not compact, then there is an unbounded $f\in C(X)$ with $Z_0\subset Z(f)$.*

(3) $(vX)_{lc}\subset X$.

(4) *Every relatively pseudocompact cozero-set of X is σ-compact.*

(5) *Every relatively pseudocompact cozero-set of X is realcompact.*

(6) *Every relatively pseudocompact cozero-set of X is nearly realcompact.*

Proof. (1)$\Rightarrow$(2) Let Z_0 and Z_1 be disjoint zero-sets of X with Z_1 not compact. Then $\mathrm{cl}_{\beta X}Z_1\subset\beta X-\mathrm{cl}_{\beta X}Z_0$ and so, since $\mathrm{cl}_{\beta X}Z_1\neq Z_1$, by (1), there is $p\in(\beta X-\mathrm{cl}_{\beta X}Z_0)\cap(\beta X-vX)$. Since $p\notin vX$, there is a zero-set Z of βX with $p\in Z\subset(\beta X-\mathrm{cl}_{\beta X}Z_0)-vX$, and there is a continuous $g:\beta X\to[0,1]$ with $\mathrm{cl}_{\beta X}Z_0\subset Z(g)$ and $Z=g^{\leftarrow}\{1\}$. Define $f\in C(X)$ by

$$f(x)=\tan\frac{g(x)\pi}{2}.$$

Since $0\leq g<1$ on X, f is defined on all of X, and so since $g(p)=1$, there is a sequence $\langle x_n: n\in\omega\rangle$ in X such that $g(x_n)\nearrow 1$. Then $f(x_n)\nearrow\infty$ and so f is unbounded. Clearly $Z_0\subset Z(f)$.

(2)$\Rightarrow$(3) Let $y\in(vX)_{lc}-X$. Then $y\in Z\subset P\subset\mathrm{int}_{vX}K\subset K\subset vX$ where K is compact and Z and P are zero-set and cozero-set neighborhoods, respectively, of y in vX. Let $Z_1=Z\cap X$ and $Z_0=X-P$. Since $y\notin X$, Z_1 is not compact and so there is an unbounded $f\in C(X)$ with $Z_0\subset Z(f)$. Then f is unbounded on $Z_0\subset P$. We can extend f to $\hat{f}$ in $C(vX)$ and $\hat{f}$ is unbounded on P. But since $P\subset K$, P is relatively pseudocompact in vX, a contradiction.

(3)$\Rightarrow$(4) Let $P=\mathrm{coz}(f)$ be a relatively pseudocompact cozero-set of X. By Proposition 1.4, $\mathrm{cl}_{vX}P$ is compact and so $vP\subset(vX)_{lc}\subset X$. Then $P=vP=\bigcup_{n\in\mathbb{N}}\hat{f}^{\leftarrow}[1/n,1]$ where $\hat{f}$ extends f to vX. Since each $\hat{f}^{\leftarrow}[1/n,1]$ is closed in $\mathrm{cl}_{vX}P$, the result follows from Proposition 1.3.

(4)$\Rightarrow$(5) and (5)$\Rightarrow$(6) are trivial.

(6)$\Rightarrow$(5) This is Proposition 1.5.

(5)$\Rightarrow$(1) Let P be a nonrealcompact cozero-set of X. By (5), P is not relatively pseudocompact and so, by Proposition 1.4, there is a decreasing sequence $\langle Z_n: n\in\omega\rangle$ of nonempty zero-sets of X with $Z_n\subset P$ for all $n\in\omega$ and $\bigcap_{n\in\omega}Z_n=\emptyset$. Then X is nearly realcompact by Theorem 1.2. $\square$

Corollary 1.8. *Every cozero-set in a nearly realcompact space is nearly realcompact.*

Proof. If P is a cozero-set in X, then every relatively pseudocompact cozero-set of P is a relatively pseudocompact cozero-set of X. $\square$

Corollary 1.9. *If X is nearly realcompact, then every relatively pseudocompact cozero-set of X is a cozero-set of βX and hence is σ-compact and locally compact.*

Proof. Let P be a relatively pseudocompact cozero-set of the nearly realcompact space X. Then $P = Q \cap X$ where Q is a cozero-set of βX. Then $Q \subset \mathrm{cl}_{\beta X}\, Q = \mathrm{cl}_{\beta X}\, P = \mathrm{cl}_{\upsilon X}\, P$ by Proposition 1.3. By Theorem 1.7(1)$\Rightarrow$(3), then, $Q \subset (\upsilon X)_{\mathrm{lc}} \subset X$. Then P is locally compact. P is σ-compact by Theorem 1.7. $\square$

Theorem 1.7(3) suggests looking at nowhere locally compact nearly realcompact spaces. The next result follows from Theorem 1.7(1)$\Rightarrow$(3).

Corollary 1.10. *The following are equivalent for any space X:*
(1) *X is nearly realcompact and nowhere locally compact.*
(2) *Every relatively pseudocompact open subset of X is empty.*

Theorem 1.11. *The following conditions are equivalent for any space X:*
(1) *X is nearly realcompact and nowhere locally compact.*
(2) *Every space that admits an open map into X is nearly realcompact.*
(3) *The product of X and any space is nearly realcompact.*
(4) *$X \times \omega_1$ is nearly realcompact.*
(5) *X is nearly realcompact, and there is a noncompact pseudocompact space Y such that $X \times Y$ is nearly realcompact.*

Proof. (1)$\Rightarrow$(2) Let $f: Y \to X$ be open and continuous, and let P be a relatively pseudocompact cozero-set in Y. Then $f^{\rightarrow}(P)$ is open and relatively pseudocompact in X, and so $f^{\rightarrow}(P) = \emptyset$ by (1) and Corollary 1.10, which means that $P = \emptyset$. Then Y is nearly realcompact by Theorem 1.7.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are trivial.

(4)$\Rightarrow$(5) If $X \times \omega_1$ is nearly realcompact, then $X \times \{0\}$, being a cozero-set of $X \times \omega_1$, is nearly realcompact by Corollary 1.8. Since $X \times \{0\}$ is homeomorphic to X, the result follows.

(5)$\Rightarrow$(1) Let $x \in X$ and suppose that K is a compact neighborhood of x in X. Let P be a cozero-set in X with $x \in P \subset K$. Now $K \times Y$ is pseudocompact [10, 9.14], and so the cozero-set $P \times Y$ of $K \times Y$ is relatively pseudocompact in $X \times Y$. By Theorem 1.7, $P \times Y$ is realcompact and so its closed subset $\{x\} \times Y$ is realcompact. But then Y is realcompact and pseudocompact, hence compact, a contradiction. $\square$

A map $f: X \to Y$ is *perfect* if it is a closed continuous surjection and $f^{\leftarrow}\{y\}$ is compact for all $y \in Y$ (that is, f has *compact fibers*). Also f is *irreducible* if f is a surjection and for all closed $F \subsetneqq X$, $f^{\to}(F) \neq Y$. It is known (see [18, 2.3]) that the image of a regular closed set under a closed irreducible map is regular closed.

A space X is *extremally disconnected* if for all open sets $U \subset X$, $\mathrm{cl}_X U$ is open. We denote the *absolute* of X by $E(X)$. Thus $E(X)$ is an extremally disconnected space which admits a perfect irreducible map onto X (see [18, 2.1]).

Theorem 1.12. *Let $\tau: Y \to X$ be closed, continuous and irreducible. Assume also that $\tau^{\leftarrow}\{x\}$ is relatively pseudocompact in Y for every $x \in X$. If A is relatively pseudocompact in X, then $\tau^{\leftarrow}(A)$ is relatively pseudocompact in Y.*

Proof. Suppose $\tau^{\leftarrow}(A)$ is not relatively pseudocompact in Y. By Proposition 1.3, there is a discrete sequence $\langle U_n : n \in \omega\rangle$ of open subsets of Y such that $U_n \cap \tau^{\leftarrow}(A) \neq \emptyset$ for all $n \in \omega$. We may assume that the U_n are regular open. Since τ is closed and irreducible, $\mathscr{F} = \{\tau^{\to} \mathrm{cl}_Y U_n : n \in \omega\}$ is a family of regular closed sets in X. We show that $\mathscr{F}$ is locally finite in X. Let $x \in X$. By Proposition 1.3, there is $N \in \omega$ such that for $n \geq N$, $\tau^{\leftarrow}\{x\} \cap U_n = \emptyset$. Let $G = X - \tau^{\to} \bigcup_{n \geq N} \mathrm{cl}_Y U_n$. G is a neighborhood of x that misses $\tau^{\to} \mathrm{cl}_Y U_n$ for all $n \geq N$. Since A is relatively pseudocompact in X, there is $N \in \omega$ such that for $n \geq N$, $A \cap \mathrm{int}_X \tau^{\to} \mathrm{cl}_Y U_n = \emptyset$. But then $U_n \cap \tau^{\leftarrow}(A) = \emptyset$, a contradiction. □

Corollary 1.13. *If $\tau: Y \to X$ is closed, continuous and irreducible, and if $\tau^{\leftarrow}\{x\}$ is relatively pseudocompact in Y for all $x \in X$, then if Y is nearly realcompact, so is X.*

Proof. Let P be a relatively pseudocompact cozero-set in X. By Theorem 1.12, $\tau^{\leftarrow}(P)$ is a relatively pseudocompact cozero-set in Y and hence is σ-compact. Then P is σ-compact and the result follows from Theorem 1.7. □

Corollary 1.14. *If $E(X)$ is nearly realcompact, then so is X.*

Example 1.15. The converse of Corollary 1.14 fails: Identify the right edge of the Tychonoff plank T with the positive integers in the reals (thus the right edge is a copy of $[0, \infty]$). The resulting space X is nearly realcompact (use Theorem 1.7(4)), but $E(T)$ is a clopen pseudocompact non-σ-compact subset of $E(X)$ and so $E(X)$ is not nearly realcompact. This example also shows that we cannot conclude in Theorem 1.7(2) that f is unbounded on Z_1. (Any $f \in C(X)$ will be bounded on the noncompact top edge.)

The following, however, is a partial converse of Corollary 1.14.

Proposition 1.16. *If X is nowhere locally compact and nearly realcompact, then so is $E(X)$.*

Proof. It suffices, by Theorem 1.7, to show that $(vE(X))_{\mathrm{lc}} = \emptyset$. Let $k: vE(X) \to vX$ extend the usual canonical map from $E(X)$ onto X, and suppose that $p \in \mathrm{int}_{vE(X)} K \subset K \subset vE(X)$ where K is compact. Then $\emptyset \neq \mathrm{int}_X(X \cap k^{\to}(K)) \subset k^{\to}(K)$, a compact set. This contradicts Corollary 1.10. □

Remark 1.17. A C-embedded zero-set of a nearly realcompact space need not be nearly realcompact: Let $X = \mathbb{Q} \times \omega_1$. X is nearly realcompact by Theorem 1.11, but $\{0\} \times \omega_1$ is a closed C-embedded zero-set that is not nearly realcompact.

2. Product, additivity and mapping theorems

We do not have a general product theorem for nearly realcompact spaces. We do have some partial results, but first we give a sum theorem.

Theorem 2.0. *If* $X = \bigoplus_{\alpha \in I} X_\alpha$ *with each* X_α *nearly realcompact, then* X *is nearly realcompact.*

Proof. Since relatively pseudocompact sets can meet only finitely many of the X_α, it suffices to note that the result follows from Theorem 1.7 when I is finite. □

Corollary 2.1. *Every discrete space is nearly realcompact.*

Proposition 2.2. *Let* $X = \bigcup_{n \in \omega} A_n$ *where each* A_n *is nearly realcompact. If*
(1) *each* A_n *is a cozero-set, or*
(2) *each* A_n *is a z-embedded zero-set,*
then X *is nearly realcompact.*

Proof. Let P be a nonrealcompact cozero-set in X. Assume $P = \mathrm{coz}(f)$ where $f \geqslant 0$. Then, by Proposition 1.1, there is $j \in \omega$ such that $f^{\leftarrow}(2^{-j}, \infty)$ is not realcompact, and there is $k \in \omega$ such that $A_k \cap f^{\leftarrow}(2^{-j}, \infty)$ is not realcompact.

(1) Suppose $A_k = \mathrm{coz}(g)$ for $g \geqslant 0$. Then there is $p \in \omega$ such that $g^{\leftarrow}(2^{-p}, \infty) \cap f^{\leftarrow}(2^{-j}, \infty)$ is not realcompact. There is, by Theorem 1.2, a decreasing sequence of nonempty zero-sets $\langle Z_n : n \in \omega \rangle$ of A_k with $Z_n \subset g^{\leftarrow}(2^{-p}, \infty) \cap f^{\leftarrow}(2^{-j}, \infty)$ and $\bigcap_{n \in \omega} Z_n = \emptyset$. Each $Z_n = Y_n \cap A_k$ where Y_n is a zero-set of X. Let $Z'_n = Y_n \cap g^{\leftarrow}[2^{-p}, \infty)$. Each Z'_n is a nonempty zero-set of X, $Z'_n \subset P$ and $\bigcap_{n \in \omega} Z'_n = \emptyset$. Then X is nearly realcompact by Theorem 1.2.

(2) Since $A_k \cap f^{\leftarrow}(2^{-j}, \infty)$ is a nonrealcompact cozero-set of A_k, there is, by Theorem 1.2, a sequence $\langle Z_n : n \in \omega \rangle$ of nonempty zero-sets of A_k, hence of X, with $Z_n \subset A_k \cap f^{\leftarrow}(2^{-j}, \infty) \subset P$ and $\bigcap_{n \in \omega} Z_n = \emptyset$. Again by Theorem 1.2, X is nearly realcompact. □

An open cover $\mathcal{U}$ of a space X is a *normal cover* of X if there is a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X with $\mathcal{U}_0 = \mathcal{U}$ and such that for all $n \in \omega$ and $U \in \mathcal{U}_{n+1}$, there is $V \in \mathcal{U}_n$ such that $\bigcup \{ W \in \mathcal{U}_{n+1} : W \cap U \neq \emptyset \} \subset V$.

Proposition 2.3. *Every space which admits a normal cover by nearly realcompact spaces is nearly realcompact.*

Proof. Let $\mathcal{U}$ be a normal cover of X by nearly realcompacts subspaces. Since $\mathcal{U}$ is normal, $\mathcal{U}$ has a σ-discrete refinement $\langle \mathcal{D}_n : n \in \omega \rangle$ of cozero-sets of X (see [9, 5.4H]). Let $D_n = \bigcup \mathscr{D}_n$. By Corollary 1.8, each $\mathscr{D}_n$ is a discrete collection of nearly realcompact spaces, and so, by Theorem 2.0, each D_n is nearly realcompact. Also each D_n is a cozero-set in X, and so X is nearly realcompact by Proposition 2.2. □

Theorem 2.4. *If X is nearly realcompact and nowhere locally compact, then*

(1) *every open subset of X is nearly realcompact;*

(2) *the union of any family of regular closed subsets of X is nearly realcompact.*

Proof. These both follow from Corollary 1.10. □

A map $f: X \to Y$ is *z-closed* if for every zero-set Z of X, $f^{\to}(Z)$ is closed in Y, and f is *z-open* if for every zero-set Z of X and for every cozero-set neighborhood H of Z in X, $f^{\to}(H)$ is a neighborhood of $\mathrm{cl}_Y f^{\to}(Z)$ in Y.

Remark 2.5. The definition and the basic results concerning z-open maps are due to Blair. They are recorded in [15, p. 177ff]. The crucial property is this: If $f: X \to Y$ is a continuous surjection, then these are equivalent: (i) f is z-open and (ii) if A and B are completely separated subsets of X, then $f^{\to}(A)$ and $Y - f^{\to}(X - B)$ are completely separated in Y [15, 15.9]. Thus z-open maps are precisely those which carry complete separation forward. The implications "open + z-closed"⇒ "z-open"⇒"open" hold and cannot be reversed. (The projection of the Tychonoff plank onto its bottom edge is z-open but not z-closed [15, 15.17(7)].)

Proposition 2.6. *Let $f: X \to Y$ be a continuous open z-closed surjection such that $f^{\leftarrow}\{y\}$ is realcompact and z-embedded in X for each $y \in Y$. If Y is nearly realcompact, then so is X.*

Proof. Let P be a relatively pseudocompact cozero-set in X and suppose that P is not realcompact. Then there is a nonrealcompact zero-set Z of X with $Z \subset P$. Since f is z-open, there is a cozero-set Q in Y with $f^{\to}(Z) \subset Q \subset f^{\to}(P)$. Now Q is relatively pseudocompact and hence realcompact by Theorem 1.7, and so $f \restriction Z: Z \to f^{\to}(Z)$ is a z-closed map with realcompact z-embedded fibers. By [15, 16.1], Z is realcompact, a contradiction. □

Question 2.7. We do not know whether the hypothesis "open and z-closed" can be reduced to "z-open" in Proposition 2.6 or whether Proposition 2.6 remains true if the fibers of f are merely nearly realcompact. We also do not know whether the countable union of nearly realcompact z-embedded subsets is nearly realcompact (see Proposition 2.2).

Corollary 2.8. *The product of a compact space and a nearly realcompact space is nearly realcompact.*

Proof. Let A be compact and Y be nearly realcompact. Then $\pi_Y: A \times Y \to Y$ is a closed surjection. Further, $\pi_Y^{\leftarrow}\{y\} = A \times \{y\}$ is realcompact and z-embedded in $A \times Y$ for each $y \in Y$. Then by Proposition 2.6, $A \times Y$ is nearly realcompact. □

Proposition 2.9. *If X and Y are nearly realcompact, and if either projection π_X or π_Y is z-open, then $X \times Y$ is nearly realcompact.*

Proof. Let P be a cozero-set in $X \times Y$ that is not nearly realcompact. By Theorem 1.7, it suffices to show that P is not relatively pseudocompact in $X \times Y$.

Choose $f \in C(X \times Y)$ with $f \geq 0$ and $P = \operatorname{coz}(f)$. By Proposition 2.2, there is $j \in \omega$ with $Q = f^{\leftarrow}(2^{-j}, \infty)$ not nearly realcompact. Let $Z = f^{\leftarrow}[2^{-j}, \infty)$ and $W = f^{\leftarrow}(2^{-(j+1)}, \infty)$. Note $Q \subset Z \subset W$.

We may assume that π_X is z-open. Then Z and $X \times Y - W$ are completely separated in $X \times Y$ and so $\pi_X^{\rightarrow}(Z)$ and $X - \pi_X^{\rightarrow}(W)$ are completely separated in X (see Remark 2.5). There are then a zero-set A and a cozero-set R of X with $\pi_X^{\rightarrow}(Z) \subset A \subset R \subset \pi_X^{\rightarrow}(W)$ and clearly $Q \subset A \times Y$. Since Q is a cozero-set of $A \times Y$ that is not nearly realcompact, then, by Corollary 1.8, $A \times Y$ is not nearly realcompact, and so, by Question 2.7, A is not compact. Further, A and $X - R$ are disjoint zero-sets of X and so, by Theorem 1.7, there is an unbounded $h \in C(X)$ with $X - R \subset Z(h)$. Let $g = h \circ \pi_X$. Clearly $g \in C(X \times Y)$ and g is unbounded on P. □

A map $f: X \to Y$ is *cozero-preserving* if $f^{\rightarrow}(P)$ is a cozero-set in Y whenever P is a cozero-set of X. If f is open and Y is perfectly normal, then f is obviously cozero-preserving. Moreover, if $X \times Y$ is z-embedded in $\beta X \times \beta Y$, then both projection maps are cozero-preserving [4, 1.1].

Proposition 2.10. *If X and Y are nearly realcompact, and if both projection maps are cozero-preserving, then $X \times Y$ is nearly realcompact.*

Proof. Let P be a cozero-set in $X \times Y$ that is not σ-compact. Since $P \subset \pi_X^{\rightarrow}(P) \times \pi_Y^{\rightarrow}(P)$, we may assume that $\pi_X^{\rightarrow}(P)$ (say) is not σ-compact and hence not relatively pseudocompact in X by Therorem 1.7. By Proposition 1.3, there is then a discrete sequence $\langle U_n \colon n \in \omega \rangle$ of open sets in X with $U_n \cap \pi_X^{\rightarrow}(P) \neq \emptyset$ for all $n \in \omega$. Then $\langle \pi_X^{\leftarrow}(U_n) \colon n \in \omega \rangle$ is a discrete sequence of open sets in $X \times Y$ with $\pi_X^{\leftarrow}(U_n) \cap P \neq \emptyset$ for all $n \in \omega$. Then, by Proposition 1.3, P is not relatively pseudocompact in $X \times Y$. By Theorem 1.7, $X \times Y$ is nearly realcompact. □

Corollary 2.11. *The product of two nearly realcompact perfectly normal spaces is nearly realcompact.*

3. Other nearly realcompact spaces

A space X is *isocompact* if every closed countably compact subspace is compact. X has *weak property* D if every infinite closed discrete subset of X has an infinite subset that is C-embedded in X. Finally X is *strongly isocompact* if X is isocompact and has weak property D. Obviously every normal isocompact space, in particular, every metric space, is strongly isocompact.

The strong isocompactness property is considerably stronger than the isocompactness property. It is important because it is both closed hereditary and productive, and so there is a "strong isocompactification" between X and βX, in fact, between X and the topological completion of X (see the proof of Corollary 3.2).

We give some characterizations of strongly isocompact spaces before discussing their relationships to nearly realcompact spaces.

Proposition 3.0. *The following conditions on X are equivalent:*

(1) *X is strongly isocompact.*

(2) *If S is relatively pseudocompact in X, then $\mathrm{cl}_X S$ is compact.*

(3) *Every relatively pseudocompact closed subset of X is compact.*

Proof. (1)$\Rightarrow$(2) Let $S \subset X$ and suppose that $\mathrm{cl}_X S$ is not compact. By (1) $\mathrm{cl}_X S$ contains an infinite closed discrete subset which contains an infinite C-embedded subset. Then $\mathrm{cl}_X S$ is not relatively pseudocompact, and so neither is S.

(2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1) Let F be a countably compact closed subset of X. Then F is relatively pseudocompact and hence compact by (3). Thus X is isocompact.

Next let D be an infinite closed discrete subset of X. Then D is not compact and hence not relatively pseudocompact by (3). By Proposition 1.3, there is a discrete sequence $\langle U_n\colon n \in \omega\rangle$ of open sets in X with $D \cap U_n \neq \emptyset$ for all $n \in \omega$. Pick $x_n \in D \cap U_n$ and note that $\langle x_n\colon n \in \omega\rangle$ is C-embedded in X. □

Corollary 3.1. *Every strongly isocompact space is nearly realcompact.*

Proof. If X is strongly isocompact, then, by Proposition 3.0, every relatively pseudocompact F_σ-set in X is σ-compact. □

Corollary 3.2. *Every topologically complete space is nearly realcompact.*

Proof. In [7, 3.1], Dykes shows that every topologically complete space satisfies Proposition 3.0(2) and hence is strongly isocompact. □

Proposition 3.3. *If $X = \prod_{\alpha \in I} X_\alpha$ with each X_α strongly isocompact, then X is strongly isocompact.*

Proof. Let F be a noncompact closed set in X. Since $F \subset \prod_{\alpha \in I} \pi_\alpha^{\rightarrow}(F)$, there is $\beta \in I$ with $\mathrm{cl}_{X_\beta} \pi_\beta^{\rightarrow}(F)$ noncompact and hence not relatively pseudocompact in X_β by Proposition 3.0. Let $f\colon X_\beta \to \mathbb{R}$ be continuous and unbounded on $\pi_\beta^{\rightarrow}(F)$. Then $f \circ \pi_\beta$ is unbounded on F, and so F is not relatively pseudocompact in X, and the result follows from Proposition 3.0. □

Corollary 3.4. *The product of strongly isocompact spaces is nearly realcompact.*

The following should be compared with Katětov's result that every paracompact space with no closed discrete measurable subset is realcompact [13, Theorem 3].

Proposition 3.5. *Every paracompact space is nearly realcompact.*

Proof. This is immediate from Corollary 3.1. □

Proposition 3.6 [MA + ¬CH]. *Every perfectly normal space is nearly realcompact.*

Proof. This follows from [16, Theorem 3]. □

Remark 3.7. Ostaszewski's space [14] is perfectly normal and countably compact but not compact and hence not nearly realcompact.

Question 3.8. Is there a ZFC example of a perfectly normal space of nonmeasurable power that is not realcompact? Or does MA + ¬CH imply that every such space is realcompact? (See [2, § 4].)

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Editorial note

This paper is based on work done by the authors in 1977 at Ohio University. The work was never published, in fact, never quite completed, but, even so, I believe that the authors consider the work important and interesting.

It was with some trepidation that I agreed (at the request of J. van Mill) to write this paper under these author's names. Both had extremely high standards for their published writing. I would therefore like the reader to understand that, while the mathematics of the paper is due to the authors, any awkwardness is entirely my own.

I would like to thank the referee who made many helpful suggestions that led to improvements in the paper. In particular, he provided the characterization in Theorem 1.7(3) as well as Example 1.15. I would also like to thank Lech T. Polkowski for several instructive conversations during the revision of the paper and John Schommer for some helpful corrections.

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